

A note on one-dependent trigonometric determinantal probability measures

Una nota sulle distribuzioni di tipo "determinantal", uno-dipendenti, trigonometriche

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Riassunto: Si fornisce una caratterizzazione delle misure di probabilità di tipo "determinantal", uno-dipendenti, trigonometriche basata sulle distribuzioni predittive. La caratterizzazione è poi applicata per dare una nuova dimostrazione che tali distribuzioni sono di tipo "two-block-factor".

Keywords: Determinantal processes, k -dependence, k -block-factors.

1. Introduction

We will start by recalling the definition of determinantal probability measure on the Borel sets of $\{0, 1\}^{\mathbf{Z}}$. Let $f : [0, 1] \rightarrow [0, 1]$ be a Lebesgue-measurable function and let $\hat{f}(k) := \int_0^1 f(x) e^{-i2\pi kx} dx$ denote its Fourier coefficients ($k \in \mathbf{Z}$). The set function

$$P^f\{x \in \{0, 1\}^{\mathbf{Z}} : x_{i_1} = 1, \dots, x_{i_k} = 1\} := \det[\hat{f}(i_r - i_s)]_{1 \leq r, s \leq k},$$

($i_1, \dots, i_k \in \mathbf{Z}$ distinct, $k \geq 1$) is a determinantal probability measure (see Lyons 2003). Determinantal processes are applied in different fields as mathematical physics, random matrix theory, representation theory and ergodic theory. For a survey see Soshnikov (2000). If the function $f : [0, 1] \rightarrow [0, 1]$ is a trigonometric function of the form

$$f(x) = \sum_{k=-m}^m a_k e^{-i2\pi kx} \quad (1)$$

with $a_k \in \mathbf{C}$, then P^f is m -dependent, according to the definition below.

Definition 1. Let $(X_i)_{i \in \mathbf{Z}}$ be a stochastic process with probability distribution P . The process $(X_i)_{i \in \mathbf{Z}}$ is called m -dependent if $(X_i)_{i < k}$ is independent of $(X_i)_{i \geq k+m}$ for all integers k .

General results are known for m -dependent trigonometric determinantal probability measures (see Soshnikov, 2000, Lyons, 2003, Shirai and Takahashi, 2003, and the references therein). In Lyons and Steif (2003), it is asked whether P^f is an $(m+1)$ -block-factor whenever f satisfies (1). We recall the definition of m -block-factor.

Definition 2. A process $(X_i)_{i \in \mathbf{Z}}$ with probability distribution P is an m -block-factor if there exists a function h of m variables and an i.i.d. process $(Y_i)_{i \in \mathbf{Z}}$ such that $(X_i)_{i \in \mathbf{Z}} \stackrel{\mathcal{D}}{=} (h(Y_i, \dots, Y_{i+m-1}))_{i \in \mathbf{Z}}$, where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution.

Every $(m + 1)$ -block-factor is trivially m -dependent. The converse is not true, in general (see Aaronson *et al.*, 1989, Aaronson *et al.*, 1992, Burton *et al.*, 1993). The question whether m -dependent determinantal probability measures are $(m + 1)$ -block-factors is still open, as far as we know. Broman (2005) has answered the question positively for $m = 1$, by finding a Borel subset A of $[0, 1]^2$ such that $(1_A(Y_i, Y_{i+1}))_{i \in \mathbb{Z}}$ has probability law P^f , when $(Y_i)_{i \in \mathbb{Z}}$ are i.i.d. random variables with uniform distribution on $[0, 1]$. In Section 2, we will provide an alternative proof of this result. Our proof is based on a predictive characterization of one-dependent trigonometric determinantal probability measures (Theorem 1).

2. Main result.

Let P^f be a one-dependent determinantal probability measure, corresponding to a trigonometric function $f(x) = a_0 + a_1 e^{i2\pi x} + a_{-1} e^{-i2\pi x}$. Notice that $a_0 \geq 0$ and $a_{-1} = \bar{a}_1$.

Being one-dependent and stationary, P^f is uniquely determined by the probabilities of the cylinder sets $C_k = \{x \in \{0, 1\}^{\mathbb{Z}} : x_1 = 1, \dots, x_k = 1\}$ ($k \geq 1$). Moreover

$$P^f(C_k) = \begin{vmatrix} a_0 & a_1 & 0 & 0 & 0 & \dots \\ \bar{a}_1 & a_0 & a_1 & 0 & 0 & \dots \\ 0 & \bar{a}_1 & a_0 & a_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix} := D_k$$

where the determinant has size $k \times k$. Let $a = a_0$ and $b = \operatorname{Re}(a_1)$. It is immediate to verify that, for every k ,

$$D_k = \begin{vmatrix} a & b & 0 & 0 & 0 & \dots \\ b & a & b & 0 & 0 & \dots \\ 0 & b & a & b & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix}.$$

In fact the equality holds for $k = 1, 2$. Moreover, for $k \geq 3$,

$$D_k = a_0 D_{k-1} - a_1 \bar{a}_1 D_{k-2} = a D_{k-1} - b^2 D_{k-2}.$$

Without loss of generality, we can therefore assume that

$$f(x) = a + b e^{i2\pi x} + b e^{-i2\pi x}, \quad (2)$$

with a and b in $\mathbf{R}$. Since f takes values in $[0, 1]$, $a \in [0, 1]$ and $|b| \leq \min(a/2, (1-a)/2)$.

The following is a characterization of one-dependent determinantal probability measures based on predictive distributions. See Cifarelli and Fortini (2005) for the proof.

Theorem 1. Let $(X_i)_{i \in \mathbb{Z}}$ be a $\{0, 1\}$ -valued stationary and one-dependent process with probability distribution P . A necessary and sufficient condition for $P = P^f$ with f as in (2) is

$$\begin{cases} P(X_1 = 1) = a \\ \operatorname{Cov}(X_{n+1}, X_{n+2} | X_1, \dots, X_n) = -b^2 \quad P - a.s. \quad (n \geq 0). \end{cases} \quad (3)$$

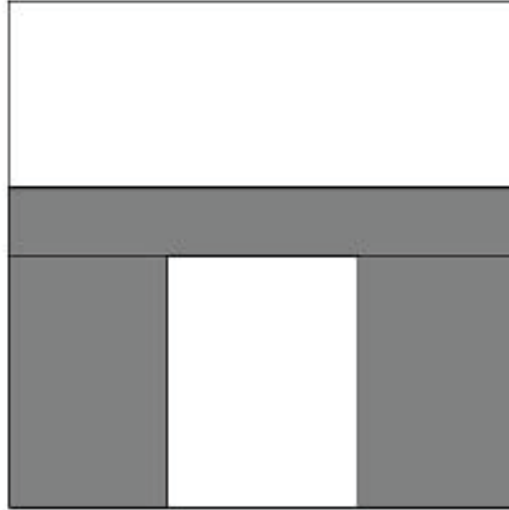
Theorem 1 can be exploited to construct a two-block-factor with law P^f . Without loss of generality, we can suppose that $0 < a \leq 1/2$. Indeed, if $(X_i)_{i \in \mathbf{Z}}$ is distributed according to P^f and f satisfies (2), then $(1 - X_i)_{i \in \mathbf{Z}}$ has probability law P^g , with $g(x) = 1 - a - be^{-i2\pi x} - be^{i2\pi x}$ (see Shirai and Takahashi 2003).

Let $(Y_i)_{i \in \mathbf{Z}}$ be i.i.d. random variables with uniform distribution on $[0, 1]$. We will prove that $(1_A(Y_i, Y_{i+1}))_{i \in \mathbf{Z}}$ satisfies (3) for a suitable choice of A , for example when A is the set

$$[0, d] \times [0, 1/2] \cup [1 - d, 1] \times [0, 1/2] \cup [0, 1] \times [1/2, 1/2 + c], \quad (4)$$

with $d = a/2 + \sqrt{a^2/4 - b^2}$ and $c = a/2 - \sqrt{a^2/4 - b^2}$

Figure 1: This figure shows A (the shaded area).



Theorem 2. For every $i \in \mathbf{Z}$, let $X_i = 1_A(Y_i, Y_{i+1})$, where A is a Borel subset of $[0, 1]^2$. If

$$\begin{cases} E(X_1) = a \\ E(X_{n+1}(X_{n+2} - a) | Y_{n+1}) = -b^2 \quad (n \geq 0), \end{cases} \quad (5)$$

then $(X_i)_{i \in \mathbf{Z}}$ satisfies (3).

PROOF. From (5) and the properties of conditional expectation, it follows that

$$\begin{aligned} & E(X_{n+1}(X_{n+2} - a) | X_1, \dots, X_n) \\ &= E(E(X_{n+1}(X_{n+2} - a) | Y_1, \dots, Y_{n+1}) | X_1, \dots, X_n) \\ &= E(E(X_{n+1}(X_{n+2} - a) | Y_{n+1}) | X_1, \dots, X_n) \\ &= -b^2 \end{aligned}$$

hold a.s., for every $n \geq 0$. This completes the proof. $\square$

Based on the Theorem 2, to construct a two-block-factor with law P^f , it is sufficient to find a Borel set A of $[0, 1]^2$ that satisfies

$$(\lambda \times \lambda)(A) = a \quad (6)$$

and

$$\int_{A_s} \lambda(A_x) dx = a\lambda(A_s) - b^2 \quad \lambda - a.s., \quad (7)$$

where $A_s = \{x \in [0, 1] : (s, x) \in A\}$ and λ is the Lebesgue measure on $[0, 1]$.

We restrict ourselves to sets A that are symmetric with respect to $\{(x, y) \in [0, 1]^2 : x = 1/2\}$ and that satisfy (6). Then (7) holds if the following conditions are verified

for every $s, t \in [0, 1/2]$, either $A_s = A_t$, or $A_s \subset A_t$ and $A_t \setminus A_s = [0, 1/2]$, or $A_t \subset A_s$ and $A_s \setminus A_t = [0, 1/2]$;

there exists s in $[0, 1/2]$ such that $\int_{A_s} \lambda(A_x) dx = a\lambda(A_s) - b^2$.

It is not difficult to find a set satisfying these conditions. The set (4) is just an example.

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